TD 9

Exercice 1 :

1. (a) $|\varphi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ $\quad B = (|+\rangle, |-\rangle)$

$$\mathbb{P}(\text{Outcome} = |+\rangle) = |\langle + | \varphi\rangle|^2 = \left| \frac{\langle 0| + \langle 1|}{\sqrt{2}} \cdot \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \right|^2$$

$$= \frac{1}{4} \left| \langle 0|0\rangle + i\langle 1|1\rangle \right|^2$$

$$= \frac{1}{4} |1+i|^2 = \frac{1}{4}\left(\sqrt{1^2+1^2}\right)^2 = \frac{1}{2}$$

$$\mathbb{P}(\text{Outcome} = |-\rangle) = 1 - \mathbb{P}(\text{Outcome} = |+\rangle) = \frac{1}{2}$$

(b) $|\varphi\rangle = \frac{|00\rangle + (1-i)|10\rangle - |11\rangle}{2}$ $\quad B = (|00\rangle, |01\rangle, |1+\rangle, |1-\rangle)$

$$\mathbb{P}(\text{Outcome} = |00\rangle) = |\langle 00|\varphi\rangle|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{2}$$

$$\mathbb{P}(\text{Outcome} = |01\rangle) = |\langle 01|\varphi\rangle|^2 = 0$$

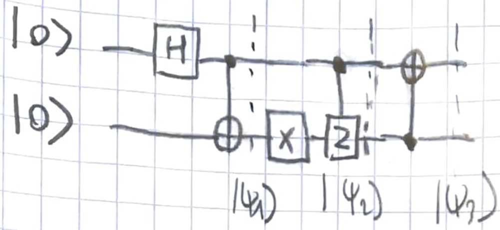
$$\mathbb{P}(\text{Outcome} = |1+\rangle) = |\langle 1+|\varphi\rangle|^2 = \left| \frac{1}{2\sqrt{2}} \left((\langle 10| + \langle 11|)((1-i)|10\rangle - |11\rangle \right) \right|^2$$

$$= \frac{1}{8} \left| (1-i)\langle 10|10\rangle - \langle 11|11\rangle \right|^2$$

$$= \frac{1}{8} |-i|^2 = \frac{1}{8}$$

So $\mathbb{P}(\text{Outcome} = |1-\rangle) = 1 - \left(\frac{1}{2} + \frac{1}{8}\right) = \frac{3}{8}$

2. (a)

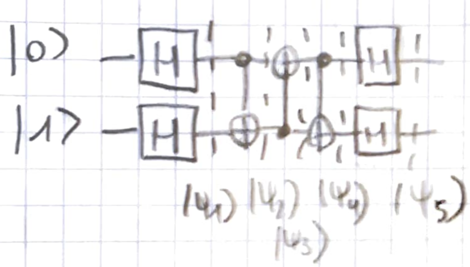


$$|\psi_1\rangle = CNOT\left(|+\rangle|0\rangle\right) = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\psi_2\rangle = CZ\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right) = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \frac{|11\rangle - |10\rangle}{\sqrt{2}} = -|1\rangle|-\rangle : \boxed{\mathbb{P}(\text{Measuring } 1) = 1}$$

(b)



$$|\psi_1\rangle = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2} = |+\rangle|-\rangle$$

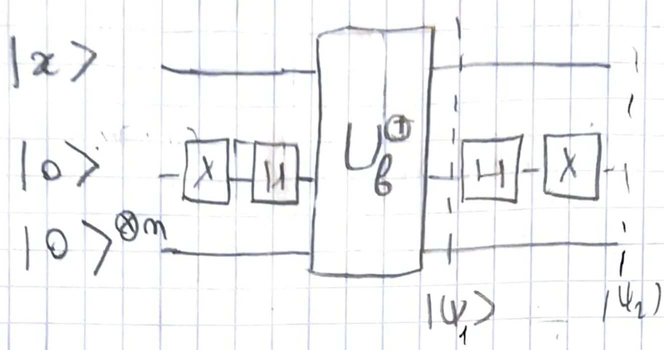
$$|\psi_2\rangle = \frac{|00\rangle - |01\rangle + |11\rangle - |10\rangle}{2}$$

$$|\psi_3\rangle = \frac{|00\rangle - |11\rangle + |01\rangle - |10\rangle}{2}$$

$$|\psi_4\rangle = \frac{|00\rangle - |10\rangle + |01\rangle - |11\rangle}{2} = |-\rangle|+\rangle$$

$$|\psi_5\rangle = |1\rangle|0\rangle \quad \text{so} \quad \mathbb{P}(\text{Measuring } 1) = 1$$

3.



$$|\psi_1\rangle = U_f^{\oplus}\left(|x\rangle|-\rangle|0\rangle^{\otimes m}\right) = \left(\frac{|x\rangle|f(x)\rangle|0\rangle^{\otimes m} + |x\rangle|1\oplus f(x)\rangle|0\rangle^{\otimes m}}{\sqrt{2}}\right)$$

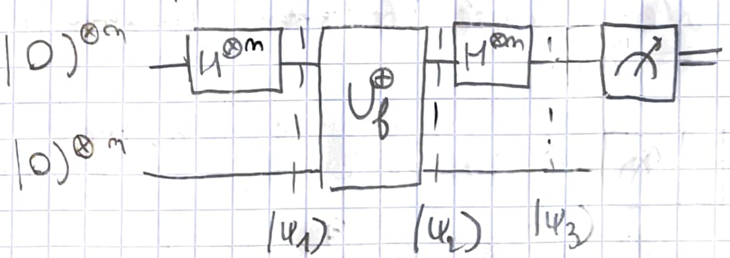
$$= \left((-1)^{f(x)}\frac{|x\rangle|0\rangle|0\rangle^{\otimes m} + |x\rangle|1\rangle|0\rangle^{\otimes m}}{\sqrt{2}}\right)$$

$$= (-1)^{f(x)}|x\rangle|-\rangle|0\rangle^{\otimes m}$$

D'où $|\psi_2\rangle = (-1)^{f(x)}|x\rangle|0\rangle|0\rangle^{\otimes m}$ $\boxed{\text{QED}}$

Exercise 2:

1. Recall Simon's algorithm



$$|\psi_1\rangle = \frac{1}{\sqrt{2^m}}\sum_{x\in\{0,1\}^m}|x\rangle\otimes|0\rangle^{\otimes m}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2^m}}\sum_{x\in\{0,1\}^m}|x\rangle\otimes|f(x)\rangle$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2^m}}\sum_{x\in\{0,1\}^m}\left(H^{\otimes m}|x\rangle\right)\otimes|f(x)\rangle$$, but $H^{\otimes m}|x\rangle = \frac{1}{\sqrt{2^m}}\sum_{y\in\{0,1\}^m}(-1)^{x\cdot y}|y\rangle$

so $$|\psi_3\rangle = \frac{1}{2^m}\sum_{x,y\in\{0,1\}^m}(-1)^{x\cdot y}|y\rangle\otimes|f(x)\rangle$$

$$= \sum_{x\in\{0,1\}^m}\left(\frac{1}{2^m}\sum_{y\in\{0,1\}^m}(-1)^{x\cdot y}|y\rangle\right)\otimes|f(x)\rangle$$

2. $\sum_{v \in V} (-1)^{v \cdot y} = \prod_{i=1}^{c} \left(1 + (-1)^{b_i \cdot y}\right)$ where $V = \underset{\mathbb{F}_2}{\operatorname{span}}(b_1, \dots b_c)$

V is a $\mathbb{F}_2$-vector space, so $V = \operatorname{span}(b_1, \dots b_c)$

$v \in V \Leftrightarrow \exists\, \varepsilon_1 \dots \varepsilon_c \in \{0,1\}$ $v = \sum_{i=1}^{c} \varepsilon_i b_i$

So $\sum_{v \in V} (-1)^{v \cdot y} = \sum_{\substack{\varepsilon_1 \dots \varepsilon_c \\ \in \{0,1\}}} (-1)^{\sum_{i=1}^{c} \varepsilon_i b_i \cdot y} = \sum_{\substack{\varepsilon_1 \dots \varepsilon_c \\ \in \{0,1\}}} \prod_{i=1}^{c} (-1)^{\varepsilon_i b_i \cdot y}$

$$= \sum_{\substack{\varepsilon_2 \dots \varepsilon_c \\ \in \{0,1\}}} \prod_{i=2}^{c} (-1)^{\varepsilon_i b_i \cdot y} \underbrace{\Big[\underbrace{(-1)^{b_1 \cdot y}}_{\varepsilon_1 = 1} + \underbrace{\underbrace{(-1)^{0 \cdot b_1 \cdot y}}_{=1}}_{\varepsilon_1 = 0} \Big]}_{= 1 + (-1)^{b_1 \cdot y}}$$

$$= \dots (\mathrm{REC}) \dots = \prod_{i=1}^{c} \left(1 + (-1)^{b_i \cdot y}\right).$$

3. We measure the last m qubits. (It does not change the result)

Then we get some $z_0 = f(x_0)$, and the remaining state is (up to normalization)

$$\sum_{v \in V} \left(\frac{1}{2^m} \sum_{y \in \{0,1\}^m} (-1)^{(x_0 + v) \cdot y} |y\rangle \right) \otimes |f(x_0)\rangle$$

But $\frac{1}{2^m} \sum_{y \in \{0,1\}^m} (-1)^{x_0 \cdot y} \left(\sum_{v \in V} (-1)^{v \cdot y} \right) |y\rangle$

$$\sum_{v \in V} (-1)^{v \cdot y} = \underbrace{\prod_{i=1}^{c} \left(1 + (-1)^{b_i \cdot y}\right)}_{\text{non-zero iff } b_i \cdot y = 0 \text{ for all } b_i} \text{ by Q 1}$$

(ie y orthogonal to V)

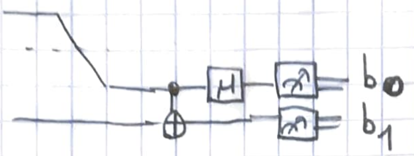
So if we measure the first m qubits, we will get y orthogonal to V

QED.

Exercise 3:

1. 2 bits

2.



After...

b_0	b_1	...CNOT	...CZ	...CNOT	... H
0	0	$\frac{\vert 00\rangle+\vert 11\rangle}{\sqrt{2}}$	/	$\vert +\rangle\vert 0\rangle$	$\vert 00\rangle$
0	1	$\frac{\vert 10\rangle+\vert 01\rangle}{\sqrt{2}}$	/	$\vert +\rangle\vert 1\rangle$	$\vert 01\rangle$
1	0	$\frac{\vert 00\rangle+\vert 11\rangle}{\sqrt{2}}$	$\frac{\vert 00\rangle-\vert 11\rangle}{\sqrt{2}}$	$\vert -\rangle\vert 0\rangle$	$\vert 10\rangle$
1	1	$\frac{\vert 10\rangle+\vert 01\rangle}{\sqrt{2}}$	$\frac{\vert 01\rangle-\vert 10\rangle}{\sqrt{2}}$	$\vert -\rangle\vert 1\rangle$	$\vert 11\rangle$

$\boxed{\text{QED}}$