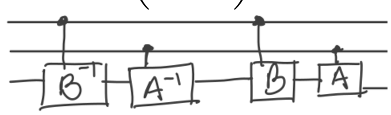
TUTORIAL 4

1 Homework 4

1. Let $U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$. Write a matrix representation of $U[1]$ and $U[2]$ for $n = 2$. For $n = 3$, write a matrix representation of CNOT$[3, 1]$.

2. Let $A = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Which 2-qubits gate can you apply on the first qubits at the end of the circuit 

to get a Toffoli gate?

2 Gate Sets for Quantum Circuits

The aim of this exercise is to prove the following theorem:

Theorem 2.1. *The set of all two-qubits unitary operators allows the realization of an arbitrary unitary operator.*

Remark. One-qubit unitaries operators are particular cases of two-qubits unitary operators.

□

Thanks to the homework, we already know that we can realize a Toffoli gate with only two-qubits unitaries. This will be a useful tool in the next parts of the proof.

2.1 Controled Unitaries

Recall that $\Lambda^k(U)$ denotes the k-controlled unitary U, which is defined by:

$$\Lambda^k(U)\left(|x_1x_2 \dots x_k\rangle \otimes |\psi\rangle\right) := \begin{cases} |x_1x_2 \dots x_k\rangle \otimes U|\psi\rangle \text{ if } x_1 \wedge x_2 \wedge \dots \wedge x_k = 1 \ , \\ |x_1x_2 \dots x_k\rangle \otimes |\psi\rangle \text{ otherwise} \ . \end{cases}$$

The aim of this part is to prove that we can realize $\Lambda^k(U)$, with U acting on one qubit, using only two-qubits unitaries.

1. Design a classical circuit that computes $x_1 \wedge x_2 \wedge \dots \wedge x_k$. What is its size? Its depth?

2. Design a quantum circuit A that computes $x_1 \wedge x_2 \wedge \dots \wedge x_k$, ie. that $A|x_1x_2 \dots x_k\rangle \otimes |0\rangle^{\otimes(N-k)} = |G(x_1x_2 \dots x_k)\rangle \otimes |x_1 \wedge x_2 \wedge \dots \wedge x_k\rangle$, with $|G(x_1x_2 \dots x_k)\rangle$ acting on $N-1$ qubits is some garbage state, using only Toffoli and NOT gates. What is its size? Its depth?

3. Design an efficient quantum circuit that computes A^{-1} efficiently. What is its size? Its depth?

4. Design a quantum circuit that computes $\Lambda^k(U)$ using only two-qubits unitaries, with the help of ancillas, ie. some circuit L such that:

$$L\left(|x_1x_2 \dots x_k\rangle \otimes |\psi\rangle \otimes |0\rangle^{\otimes(N-1-k)}\right) = \left(\Lambda^k(U)\left(|x_1x_2 \dots x_k\rangle \otimes |\psi\rangle\right)\right) \otimes |0\rangle^{\otimes(N-1-k)} \ .$$

2.2 Almost Diagonal Unitaries

The aim of this part is to show that unitaries U on $\mathbb{C}^{2^n}$ of the form Diag $\left(1,\ldots,1,\begin{pmatrix} a & b \\ c & d \end{pmatrix},1,\ldots,1\right)$ can be realized using only two-qubits unitaries.

1. What can you say about $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$?
2. Write $\Lambda^{n-1}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ in matrix form. Show that there exists a permutation matrix P such that:
$$U = P^{-1}\Lambda^{n-1}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) P\ .$$
3. Design a quantum circuit that computes P and another that computes P^{-1} using only two-qubits unitaries. What are their sizes? Their depths?
4. Design a quantum circuit that computes Diag $\left(1,\ldots,1,\begin{pmatrix} a & b \\ c & d \end{pmatrix},1,\ldots,1\right)$. What is its size? Its depth?

2.3 General form of an Arbitrary Unitary

Recall the following lemma seen during last lecture:

Lemma 2.2. *Any unitary operator U on $\mathbb{C}^M$ can be written as a product of $\mathcal{O}(M^2)$ unitary matrices of the form Diag* $\left(1,\ldots,1,\begin{pmatrix} a & b \\ c & d \end{pmatrix},1,\ldots,1\right)$.

1. Prove theorem 2.1, using ancillas. What is the size of the circuit? Its depth?

3 Quantum 1-Machine

You have a device that outputs only $|0\rangle$. You can use this device several times, and measure in any basis. How many calls to the device do you need to get a $|1\rangle$?

4 Simon's Problem Generalized

Consider a function $f : \{0,1\}^n \to \{0,1\}^n$ with the promise that there exists a vector subspace V of $\{0,1\}^n$ (seen as a vector space over $\mathbb{F}_2$) such that:

$$\forall x, y \in \{0,1\}^n,\ f(y) = f(x) \Leftrightarrow \exists v \in V, x = y + v\ .$$

Show that one run of Simon's algorithm output $x \in \{0,1\}^n$ such that x is orthogonal to V ($\forall y \in V, x \cdot y = 0 \text{ MOD } 2$).

Remark: the usual version of Simon's Problem is when $V = \{0, a\}$ for some $a \in \{0,1\}^n$.