TD 4

1. HW 4

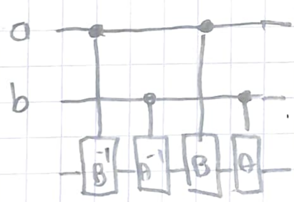
1. $U[1] = U \otimes I_2 = \begin{array}{cc} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} & \\ \begin{pmatrix} U_{00} & 0 & U_{01} & 0 \\ 0 & U_{00} & 0 & U_{01} \\ U_{10} & 0 & U_{11} & 0 \\ 0 & U_{10} & 0 & U_{11} \end{pmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$

$U[2] = I_2 \otimes U = \begin{array}{cc} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} & \\ \begin{pmatrix} U_{00} & U_{01} & 0 & 0 \\ U_{10} & U_{11} & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$

$CNOT[3,1] = \begin{array}{cc} \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} \end{array}$

=

Recall $CNOT = \begin{array}{cc} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$

2. 

If $a = 0$ or $b = 1$, then the unitary on the third qubit is $I_2 = B^{-1}B = A^{-1}A$

So the only case of interest is $a = b = 1$:
The global unitary U on the 3^{rd} qubit is $U = B^{-1}A^{-1}BA$

First $B^{-1} = B^{\dagger} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $A^{-1} = A^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}$

Then $B^{-1}A^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ and $BA = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

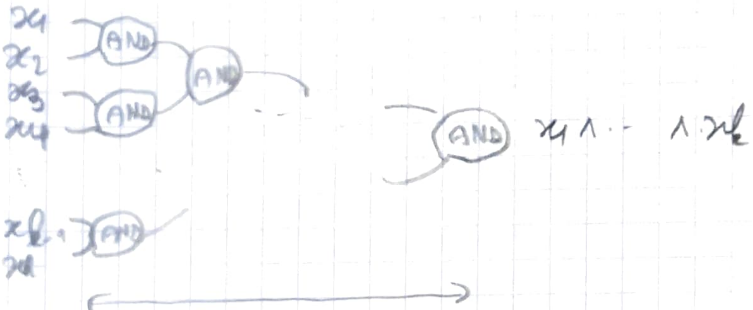
So $U = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is the application of Toffoli up to a phase $-i$ for that case

So to construct V on the first two qubits, it should be the identity if $a = 0$ or $b = 0$, and cancel the phase $-i$ if $a = b = 1$

$$V = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -i \end{pmatrix}$$ will work since $-i^2 = 1$

2. Gate Sets for Quantum Circuits:

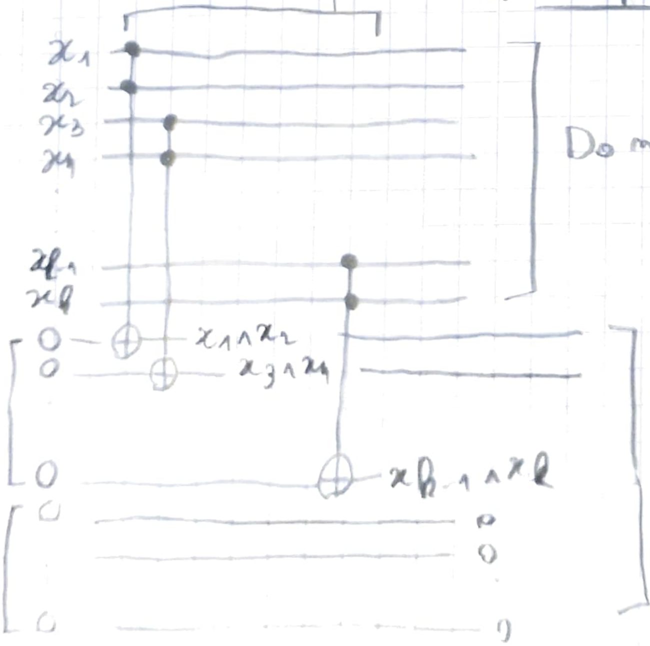
$\boxed{2.1}$ - 1



Depth = $\lceil \log_2 k \rceil$ Size = $k-1$ AND gates

Parallel computation so 1 depth

2. A:



Do not use any more

Same problem with size $k/2$

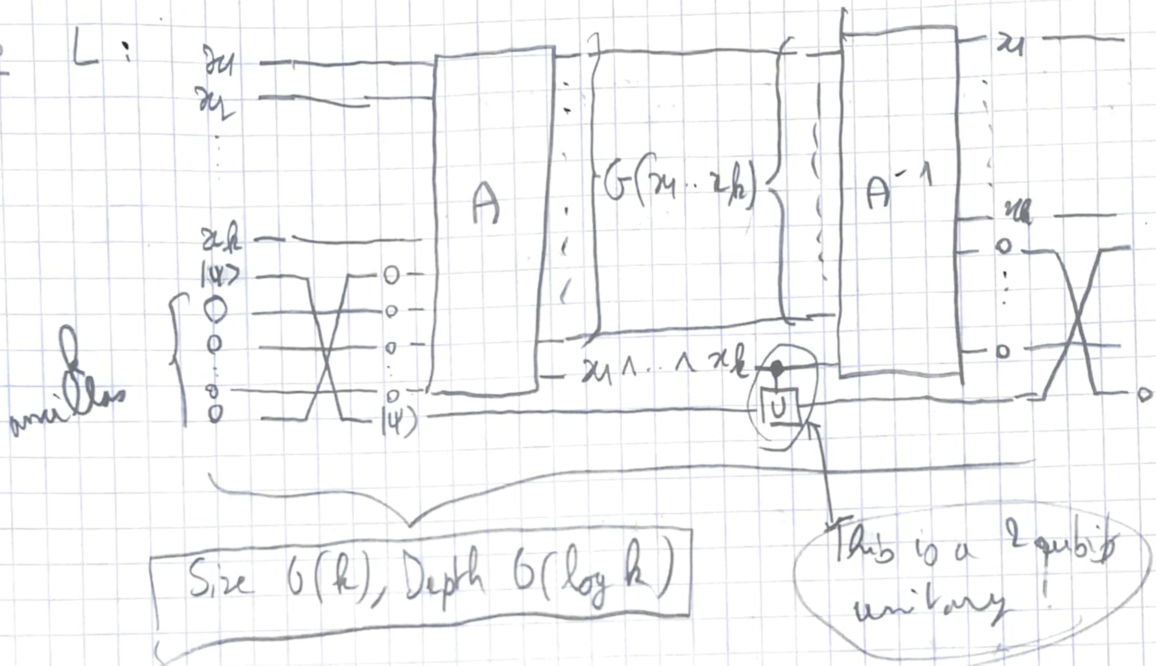
Repeat this $\lceil \log_2 k \rceil$ times: get same size and depth as the classical case using only Toffoli gates

3. The same minor circuit works since

Try it on the linear circuit
(depth $O(k)$)

4. A and A^{-1} can be computed in size $O(k)$ and depth $O(\log k)$ using only 2 qubits gates since Toffoli can be replaced by 5 1-qubits gate (HW4)

So L:



$\boxed{2.2}$ 1. $U^\dagger U = I_2 \Rightarrow \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}^\dagger = I_2$ (C) $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ is unitary

2. $$\Lambda^{m-1}\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{array}{cccc} |00..0\rangle & |1111011\rangle & |11\cdots10\rangle & |11\cdots11\rangle \\ \end{array}$$
$$\begin{pmatrix} 1 & & & 0 & \\ & \ddots & & & \\ & & 1 & & \\ 0 & & & \boxed{\begin{matrix} a & b \\ c & d \end{matrix}} & \end{pmatrix} \begin{array}{c} \\ \\ \\ |111..10\rangle \\ |111..11\rangle \end{array}$$

1st qubit, last qubit

CRUCIAL to have a diagonal coefficient

$\rightarrow$ Just take P_σ where $\sigma = (i{+}1 \;\; m)(i \;\; m-1)$

3. Just two transpositions that can be done as 2 swaps
P^{-1} is the same as P in this particular case. (if $i+1 \neq n-1$
otherwise just mirror)

So size and depth in $O(1)$

RQ in general for a permutation, need $(n-1)$ transpositions
that do not commute!

4. Combine $U = \underbrace{P^{-1} \; \Lambda^{n-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \; P}_{\text{Size } O(n),\ \text{Depth } O(\log n)}$ constant

Size $O(n)$
Depth $O(\log n)$

$\boxed{2.3}$ 1. $M = 2^n$

Just combine (2.2.4) $O(M^2)$ times

$\hookrightarrow$ get size $O(n(2^n)^2)$ Depth $O(\log n\,(2^n)^2)$