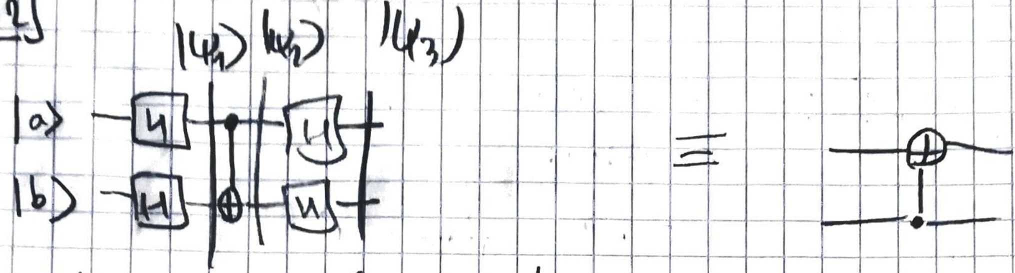
# Détails:

1. $\boxed{\text{HW2}}$

1.




$$|\psi_1\rangle = \frac{1}{2}\left(|0\rangle + (-1)^a|1\rangle\right)\left(|0\rangle + (-1)^b|1\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle + (-1)^b|01\rangle + (-1)^a|10\rangle + (-1)^{a+b}|11\rangle\right)$$

$$|\psi_2\rangle = \frac{1}{2}\left(|00\rangle + (-1)^b|01\rangle + (-1)^{a+b}|10\rangle + (-1)^a|11\rangle\right)$$

$$|\psi_3\rangle = \frac{1}{2}\left(|{+}{+}\rangle + (-1)^b|{+}{-}\rangle + (-1)^{a+b}|{-}{+}\rangle + (-1)^a|{-}{-}\rangle\right)$$

$ab = 00$ $\quad |\psi_3\rangle = \frac{1}{2}\left(|+\rangle+|-\rangle\right)\left(|+\rangle+|-\rangle\right) = |00\rangle$

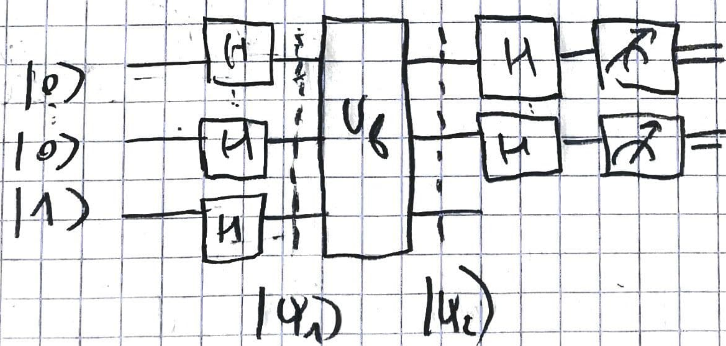
$10$ $\quad |\psi_3\rangle = \frac{1}{2}\left(|+\rangle-|-\rangle\right)\left(|+\rangle+|-\rangle\right) = |10\rangle$

$01$ $\quad |\psi_3\rangle = \frac{1}{2}\left(|+\rangle-|-\rangle\right)\left(|+\rangle-|-\rangle\right) = |11\rangle$

$11$ $\quad |\psi_3\rangle = \frac{1}{2}\left(|+\rangle+|-\rangle\right)\left(|+\rangle-|-\rangle\right) = |01\rangle$

QED

2. Circuit:




If output $= (00.00)$ constant
Else balanced

$$|\psi_1\rangle = \left(2^{-n/2} \sum_{x\in\{0,1\}^n} |x\rangle\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\psi_2\rangle = 2^{-\frac{n+1}{2}} \sum_x |x\rangle \, (\underbrace{|f(x)\rangle - |1 \oplus f(x)\rangle}_{= (-1)^{f(x)} (|0\rangle - |1\rangle)})$$

$$= \underbrace{\left(2^{-n/2} \sum_x (-1)^{f(x)} |x\rangle\right)}_{\text{State on first } n \text{ qubits}} \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\psi_3\rangle \,(n \text{ qubits}) = H^{\otimes n} |x\rangle = 2^{-n/2} \sum_y (-1)^{x \cdot y} |y\rangle$$

So $|\psi_3\rangle = 2^{-n} \sum_{x\,y} (-1)^{f(x)} (-1)^{x\cdot y} |y\rangle$

So $\text{Proba}(00\ldots0) = \left| 2^{-n} \sum_x (-1)^{f(x)} \right|$

$$= \begin{cases} 1 & \text{if } f \text{ is constant} \\ 0 & \text{if } f \text{ balanced} \end{cases}$$

- Classicaly : $2^{n/2} + 1$ calls to distinguish
- With error
  - Take an element $x$ uniformly in $\{0,1\}^n$, store $f(x)$ and repeat. Take in all $k$ elements.
  - If there exists 2 differents values $f(x) = 0$ and $f(a) = 1$ return "Balanced" # always correct here
  
    Else return "Constant"

$\mathbb{P}(\text{error}) = \mathbb{P}(\text{output} = \text{constant} \mid f \text{ balanced}) = \mathbb{P}(\text{ele}2 = \text{ele}1 \mid \text{balanced}) \ldots \mathbb{P}(\text{ele}k = \text{ele}1 \mid \text{balanced}) = \left(\frac{1}{2}\right)^{k-1}$ (independent events)

**2.2.** $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|\varphi\rangle + |1\rangle|\psi\rangle\right)$

$$H\otimes I\,|\Psi\rangle = \frac{1}{2}\Big[\,|0\rangle\big[|\varphi\rangle + |\psi\rangle\big] + |1\rangle\big[|\varphi\rangle - |\psi\rangle\big]\Big]$$

Partial measurement on first qubit

$$\mathbb{P}(\sigma = 1) = \left\|\tfrac{1}{2}\left(|\varphi\rangle - |\psi\rangle\right)\right\|^2$$

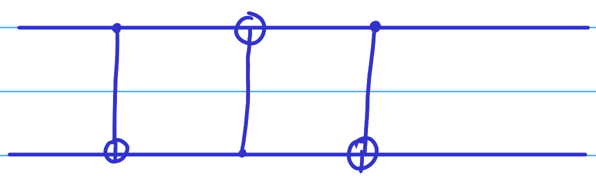
$$= \frac{1}{4}\left(\||\varphi\rangle\|^2 + \||\psi\rangle\|^2 - \left(\langle\psi|\varphi\rangle + \langle\varphi|\psi\rangle\right)\right)$$

$$= \frac{1}{4}\left(2 - \left(\langle\varphi|\psi\rangle + \overline{\langle\varphi|\psi\rangle}\right)\right)$$

$$= \frac{1}{4}\left(2 - 2\,\mathrm{Real}\left(\langle\varphi|\psi\rangle\right)\right)$$

$$\boxed{= \frac{1}{2}\left(1 - \mathrm{Real}\left(\langle\varphi|\psi\rangle\right)\right)}$$

**2.3.** SWAP Gate :



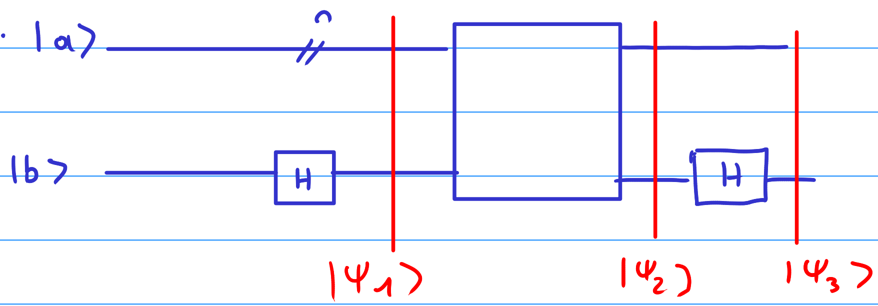
Clearly $\quad CNOT\,|b_0 b_1\rangle = |b_0\rangle|b_0\oplus b_1\rangle$

$CNOT' \hookrightarrow |b_0\oplus b_0\oplus b_1\rangle|b_0\oplus b_1\rangle$

$= |b_1\rangle|b_0\oplus b_1\rangle$

$CNOT \hookrightarrow |b_1\rangle|b_0\oplus b_1\oplus b_1\rangle = |b_1\rangle|b_0\rangle$

2.




$$|\psi_1\rangle = |a\rangle\otimes\left(\frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^b|1\rangle\right)\right)$$

$$|\psi_2\rangle = |a\rangle\otimes\left(\frac{1}{\sqrt{2}}\left(|f(a)\rangle + (-1)^b|\overline{f(a)}\rangle\right)\right)$$

$$= \begin{cases} \text{if } b=0 & |a\rangle\otimes|+\rangle \\ \text{if } b=1 & (-1)^{f(a)}|a\rangle\otimes|-\rangle \end{cases}$$

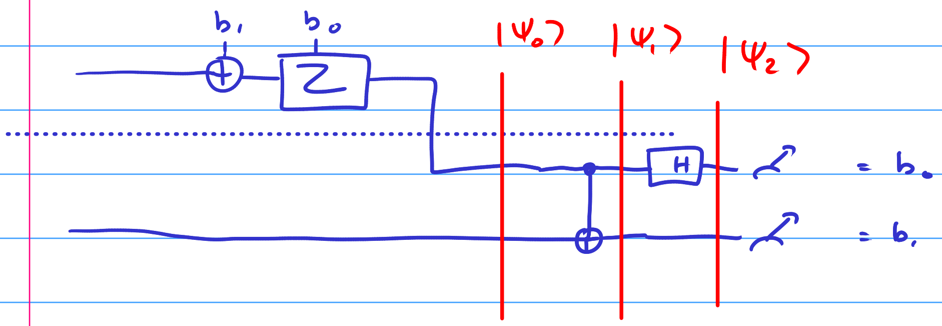
$$|\psi_3\rangle = \begin{cases} \text{if } b=0 & |a\rangle|0\rangle \\ \text{if } b=1 & (-1)^{f(a)}|a\rangle|1\rangle \end{cases} = (-1)^{b \cdot f(a)}|a\rangle|b\rangle$$

□

3. 1. Informal, Bob has to send 2 bits

2.

Solution circuit




| $b_0$ | $b_1$ | $\|\psi_0\rangle$ | $\|\psi_1\rangle$ | $\|\psi_2\rangle$ |
|---|---|---|---|---|
| 0 | 0 | $\frac{1}{\sqrt{2}}[\|00\rangle + \|11\rangle]$ | $\frac{1}{\sqrt{2}}[\|00\rangle + \|10\rangle] = \|+\rangle\|0\rangle$ | $\|00\rangle$ |
| 0 | 1 | $\frac{1}{\sqrt{2}}[\|10\rangle + \|01\rangle]$ | $\frac{1}{\sqrt{2}}[\|11\rangle + \|01\rangle] = \|+\rangle\|1\rangle$ | $\|01\rangle$ |
| 1 | 0 | $\frac{1}{\sqrt{2}}[\|00\rangle - \|11\rangle]$ | $\frac{1}{\sqrt{2}}[\|00\rangle - \|10\rangle] = \|-\rangle\|0\rangle$ | $\|10\rangle$ |
| 1 | 1 | $\frac{1}{\sqrt{2}}[-\|10\rangle + \|01\rangle]$ | $\frac{1}{\sqrt{2}}[-\|11\rangle + \|01\rangle] = \|-\rangle\|1\rangle$ | $\|11\rangle$ |

□