HW

1. Error: $Z \otimes I^{n-1} = A$, $B = I$

$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0^n\rangle + |1^n\rangle)$ $\qquad$ $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0^n\rangle - |1^n\rangle)$

We have $\langle \psi_1 | \psi_2 \rangle = 0$

but $A|\psi_1\rangle = |\psi_2\rangle$

So $\langle \psi_1 | A^+ B | \psi_2 \rangle = 1$

2. $\mathcal{C}_S = \{ |\psi\rangle \,/\, g \cdot |\psi\rangle = |\psi\rangle \ \forall g \in S \}$

Let $g \in S$, $\mathcal{C}_g = \{ |\psi\rangle \,/\, g \cdot |\psi\rangle = |\psi\rangle \}$
$= \ker(g - id)$ is a vector space

$\mathcal{C}_S = \bigcap_{g \in S} \mathcal{C}_g$ v. space $\square$

Ex 1

1.
$$\phi_n(A, B) = \sum_{i=1}^{n} (AB^+)_{ii}$$
$$= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B^+_{ki}$$
$$= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} \overline{B_{ik}}$$

$\Rightarrow$ ℓ_2 scalar product over $\mathbb{C}^{n \times n}$!

2. $\phi_{nm}(A \otimes B, A' \otimes B') = Tr(AA'^+ \otimes BB'^+)$

We can compute $Tr(A \otimes B) = Tr(A) \cdot Tr(B)$

Hence $\phi_{nm}(A \otimes B, A' \otimes B') = Tr(AA'^+) \cdot Tr(BB'^+)$ $\square$

3. $\langle I, X\rangle = 0 = \langle I, Y\rangle = \langle I, Z\rangle$
$\langle X, Y\rangle = \langle X, Z\rangle = 0$
$\langle Y, Z\rangle = 0$
check easy

So $\{I, X, Y, Z\}$ is an orthogonal family of size 4 and it is in $\mathbb{C}^{2\times 2}$ of dim 4 $\Rightarrow$ it is a basis.

4. By 2., $\langle \sigma(w), \sigma(w')\rangle = \begin{cases} 0 & \text{iff } w \neq w' \\ 1 & \text{else} \end{cases}$

So $\left\{\sigma(w),\ w \in \{I, X, Y, Z\}^n\right\}$ is O.G. of size 4^n in $\mathbb{C}^{2^n \times 2^n}$

$\square$

5. The set $\underset{E}{\overset{\shortparallel}{\sigma(\{w \mid |w| \leq t\})}}$ is orthogonal.
$\forall\ |w| \leq t$, $\sigma(w) \subset \mathcal{E}[\text{support}(w)] \subset \mathcal{E}(n,t)$

So $E \subseteq \mathcal{E}(n,t)$

By 2., $\mathcal{E}[A] = \text{span}(\sigma(w \mid w_i \neq I \Leftrightarrow i \in A))$
so for any $A \subseteq \{1, \dots, n\}$ of size $\leq t$,

$\mathcal{E}[A] \subseteq E \Rightarrow \mathcal{E}(n,t) \subseteq E$

$\square$

TD 14

3. Stabilizer Codes:

1. Z_1Z_2 : stabilizes $|000\rangle$, $|001\rangle$, $|110\rangle$, $|111\rangle$

symm $\Big($ Z_1Z_3 : stabilizes $|000\rangle$, $|010\rangle$, $|101\rangle$, $|111\rangle$

Z_2Z_3 : stabilizes $|000\rangle$, $|100\rangle$, $|011\rangle$, $|111\rangle$

So only $|000\rangle$ and $|111\rangle$ in common

So $C_S = \text{span}(|000\rangle, |111\rangle)$

<u>INDEED</u> : If $Z_1Z_2 |\psi\rangle = |\psi\rangle$ with $|\psi\rangle = \sum_{i,j,k=0}^{1} \alpha_{ijk} |ijk\rangle$

<u>Then</u> $Z_1Z_2 |\psi\rangle = \sum_{i,j,k=0}^{1} (-1)^{i+j} \alpha_{ijk} |ijk\rangle$

<u>So</u> if $Z_1Z_2|\psi\rangle = |\psi\rangle$

<u>then</u> $\forall i,j,k = 0,1 \quad (-1)^{i+j} \alpha_{ijk} = \alpha_{ijk}$

$\boxed{\forall ijk = 0,1, \; i+j = 1 \Rightarrow \alpha_{ijk} = 0}$ CNS on $C_{Z_1Z_2}$

So $\boxed{C_{Z_1Z_2} = \text{span}(|000\rangle, |001\rangle, |110\rangle, |111\rangle)}$

and $C_S = C_{Z_1Z_2} \cap C_{Z_1Z_3} \cap C_{Z_2Z_3}$

2. . Easy direction : since $h \in H \Rightarrow h \in S$

then $|\psi\rangle \in C_S \Rightarrow \forall h \in H \;\; h|\psi\rangle = |\psi\rangle$

. <u>Other direction</u> : $\forall h \in H, \; h|\psi\rangle = |\psi\rangle$

So for $g \in S$, $g = h_1 \dots h_l$ with $h_i \in H$

so $g|\psi\rangle = h_1 \dots h_l |\psi\rangle = h_1 \dots h_{l-1} |\psi\rangle = \dots = |\psi\rangle$

So: $\forall h \in H,\ h|\psi\rangle = |\psi\rangle \Rightarrow \forall g \in S\ g|\psi\rangle = |\psi\rangle = |\psi\rangle \in C_S$ $\blacksquare$.

3. If $-I \in S$, then if $|\psi\rangle \in C_S$, by definition $\forall h \in S,\ h|\psi\rangle = |\psi\rangle$

so in particular $-I|\psi\rangle = |\psi\rangle$ ie. $-|\psi\rangle = |\psi\rangle$ ie $2|\psi\rangle = 0$

ie. $\boxed{|\psi\rangle = 0}$

So $C_S \subset \{0\}$ ie $\boxed{C_S = \{0\}}$

4. Lemma: **Key point**

$\boxed{\text{In } G_1, \text{ for } g, h \in G_1, \text{ either } gh = hg \text{ (and thus in } G_n) \quad \boxed{\text{OR}}\ gh = -hg}$

Thus: if $gh = -hg$

then $gh \times (hg)^{-1} = -(hg)(hg)^{-1} = \boxed{-I}$

so by 3., we get $\boxed{C_S = \{0\}}$

D $G_1 = \langle -I, iI, X, Y, Z \rangle$ $\quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

→ With $-I$ & iI; X, Y & Z commute

→
- $XY = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iZ$
- $YX = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -iZ = -XY$
- $XZ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -iY$
- $ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = iY = -XZ$
- $YZ = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = iX$
- $ZY = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -iX = -YZ$

QED Lemma

→ THEN for G_n, $(g_1 \otimes \cdots \otimes g_n)(h_1 \otimes \cdots \otimes h_n) = (g_1 h_1) \otimes \cdots \otimes (g_n h_n)$

$= (-1)^k (h_1 g_1) \otimes \cdots \otimes (h_n g_n) = (-1)^k (h_1 \otimes \cdots h_n)(g_1 \otimes \cdots g_n)$

number of non-commuting elements.

$\boxed{\text{So TRUE for } G_n \text{ ALSO}}$

5. Dimension 2^1, so we find $9-1=8$ independent, noncommuting elements that stabilize Shor Code will be enough!

$\to Z_1 Z_2\ ,\ Z_2 Z_3$
$\to Z_4 Z_5\ ,\ Z_5 Z_6$
$\to Z_7 Z_8\ ,\ Z_8 Z_9$

$|000\rangle + |111\rangle \to |000\rangle + |111\rangle$
$|000\rangle - |111\rangle \to |000\rangle - |111\rangle$

6 Ok commute

Rk $Z_1 Z_3 = Z_1 Z_2 \times Z_2 Z_3$ since $\boxed{Z^2 = I}$

so not indep!

Need 2 more: $X_1 X_2 X_3 X_4 X_5 X_6$

IDEM $X_4 X_5 X_6 X_7 X_8 X_9$

Ok commute

$|000\rangle + |111\rangle \to |000\rangle + |111\rangle$
$|000\rangle - |111\rangle \to -|000\rangle + |111\rangle$

But apply it twice: 3 first and 3 second qubits

makes $-$ disappear

$X_1 X_2 X_3 X_4 X_5 X_6$ commute? with $Z_1 Z_2$

$\to X_1 X_2 \cdot Z_1 Z_2 = (XZ) \otimes (XZ) \otimes \cdots$
$= (-ZX) \otimes (-ZX) \otimes \cdots$
happens twice! $= (ZX) \otimes (ZX) \otimes \cdots$
$= Z_1 Z_2 \cdot X_1 X_2$!

$\boxed{\text{Ok commute}}$

$\boxed{\text{QED}}$.