TD 12

2. Shannon Channel Coding Theorem :

2.1 1. $D(y^n) = \underset{(\epsilon)}{\arg\max}_{s} \underbrace{W^n(y^n | E(s))}$

Most likely s that outputs y^n

2. $P_{err,s} = 1 - \sum_{\substack{y^n \in Y^n \\ D(y^n)=s}} W^n(y^n | E(s))$

$$= \sum_{y^n \in Y^n} W^n(y^n | E(s)) \, \mathbb{1}_{D(y^n) \neq s}$$

3. $\mathbb{1}_{D(y^n)\neq s} \leq \mathbb{1}_{\left(\underbrace{W^n(y^n|E(s)) < \alpha(n,y^n)}_{\text{either } s \text{ is not in } 1^{st} \text{ case}}\right) \vee \left(\underbrace{\exists s' \neq s \; W^n(y^n|E(s')) \geq \alpha(n,y^n)}_{s \text{ is not the only } s' \text{ in } 1^{st} \text{ case}}\right)}$

$$\leq \mathbb{1}_{W^n(y^n|E(s)) < \alpha(n,y^n)} + \mathbb{1}_{\exists s' \neq s \; W^n(y^n|E(s')) \geq \alpha(n,y^n)}$$

$$\leq \mathbb{1}_{W^n(y^n|E(s)) < \alpha(n,y^n)} + \sum_{s' \neq s} \mathbb{1}_{W^n(y^n|E(s')) \geq \alpha(n,y^n)}$$

Thus, taking the sum over $y^n \in Y^n$, one gets $P_{err,s} \leq P^1_{err,s} + P^2_{err,s}$

2.2 1. If $\mathbb{E}_E[P_{err}] \underset{n \to +\infty}{\to} 0$

It means that for any n, there exists E^n such that $\underbrace{P_{err}^{E^n}}_{\text{property of expectancy}} \leq \mathbb{E}_E[P_{err}]$

So in particular $P_{err}^{E^n} \underset{n \to +\infty}{\to} 0$ QED

BR : $\mathbb{P}\left(P_{err}^{E} \leq \underbrace{2\mathbb{E}_E[P_{err}]}_{\to 0}\right) \geq \frac{1}{2}$ by Markov inequality, which gives a good enough scheme E.

2 @ $\mathbb{E}_\mathcal{E}[P'_{em,0}] = \sum_{x^n} P_{X^n}(x^n)\, P'_{em,0,\mathcal{E}(x)=x^n}$ by definition of random $\mathcal{E}(\cdot)$

$= \sum_{x^n y^n} P_{X^n}(x^n)\, \hat{W}(y^n|x^n)\, \mathbb{1}_{W^n(y^n|x^n) \le K(m,\varepsilon) P_{Y^n}(y^n)}$

$= P_{X^n Y^n}(x^n, y^n)$ by def w/ a

But

So with $X^n Y^n = (X_1 Y_1, X_2 Y_2 \dots X_n Y_n)$ with $X_i Y_i \sim P_{XY}$

we have that $\mathbb{E}_\mathcal{E}[P'_{em,0}] = \mathbb{P}\left(\prod_{i=1}^n W(Y_i|X_i) < K(m,\varepsilon) \prod_{i=1}^n P_Y(Y_i) \right)$

(b) $\mathbb{P}\left(\prod_{i=1}^n W(Y_i|X_i) < K(m,\varepsilon) \prod_{i=1}^n P_Y(Y_i) \right)$

$= \mathbb{P}\left(\sum_{i=1}^n \log(W(Y_i|X_i)) < \log K(m,\varepsilon) + \sum_{i=1}^n \log P_Y(Y_i) \right)$

$= \mathbb{P}\left(\sum_{i=1}^n \underbrace{\log\left(\frac{P_{XY}(X_i,Y_i)}{P_X(X_i)\, P_Y(Y_i)} \right)}_{i_{XY}(X_i,Y_i)} < \log K(m,\varepsilon) \right)$

(c) $\mathbb{E}[i_{XY}(X_i,Y_i)] = \sum_{xy} P_{XY}(xy)\, i_{XY}(x,y) = I(X:Y) = C(W)$

since P_X has been chosen to make $I(X:Y)$ achieve $C(W)$

(d) $\lim_{n \to +\infty} \mathbb{P}\left(\left| \frac{1}{n} \sum_{i=1}^n i_{XY}(X_i,Y_i) - C(W) \right| < \varepsilon \right) = 1$

So $\mathbb{E}_\mathcal{E}[P'_{em,0}] = \mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n i_{XY}(X_i,Y_i) - C(W) < \frac{1}{n} \log K(m,\varepsilon) - C(W) \right)$

$\xrightarrow[n \to +\infty]{} 0$ if $\frac{1}{n} \log K(m,\varepsilon) - C(W) \le -\varepsilon$

ie. $\boxed{K(n,\varepsilon) \le e^{n(C(W)-\varepsilon)}}$ for some $\varepsilon > 0$

3. $\mathbb{E}_{\mathcal{E}}[P^{2}_{e_n,\sigma}] = \sum_{\sigma' \neq \sigma} \mathbb{E}_{\mathcal{E}}\left[\sum_{y^n} W^n(y^n | \mathcal{C}(\sigma)) \, \mathbb{1}_{W^n(y^n | \mathcal{C}(\sigma')) \geq \dots}\right]$

one term

$\leq 2^{Rn} \sum_{x^n, x'^n, y^n} P_{X^n}(x^n) P_{X^n}(x'^n) W^n(y^n | x^n) \, \mathbb{1}_{W^n(y^n | x'^n) \geq K(n,\varepsilon) P_{Y^n}(y^n)}$

size of σ' space — does not depend on x^n

$\leq 2^{Rn} \sum_{x'^n, y^n} P_{Y^n}(y^n) P_{X^n}(x'^n) \, \mathbb{1}_{W^n(y^n | x'^n) \geq K(n,\varepsilon) P_{Y^n}(y^n)}$

use [illegible] by

$\leq 2^{Rn} \sum_{x'^n, y^n} \frac{W(y^n | x'^n)}{K(n,\varepsilon)} P_{X^n}(x'^n) = \frac{2^{Rn}}{K(n,\varepsilon)}$

So it is sufficient that $\boxed{\frac{2^{Rn}}{K(n,\varepsilon)} \xrightarrow[n \to +\infty]{} 0}$ to conclude

4. $R < C(W)$, take $\varepsilon = \frac{C(W) - R}{2}$ and $K(n,\varepsilon) = 2^{(R+\varepsilon)n}$

Then $K(n,\varepsilon)$ fulfills both condition

Thus $\mathbb{E}_{\mathcal{E}}[P_{e_n}] = \frac{1}{2^{Rn}} \sum_{\sigma} \mathbb{E}[P_{e_n,\sigma}]$

$\leq \frac{1}{2^{Rn}} \sum_{\sigma} \underbrace{\mathbb{E}[P^{1}_{e_n,\sigma}]}_{\xrightarrow[n \to +\infty]{} 0} + \frac{1}{2^{Rn}} \sum_{\sigma} \underbrace{\mathbb{E}[P^{2}_{e_n,\sigma}]}_{\xrightarrow[n \to +\infty]{} 0}$

So $\mathbb{E}_{\mathcal{E}}[P_{e_n}] \xrightarrow[n \to +\infty]{} 0$, and we conclude with [Q1] ∎

2.3 1 $P_X = (p, 1-p)$ for some $p \in [0,1]$

Then $P_Y(0) = (1-p)(1-\beta) + p\beta$

$P_Y(1) = p(1-\beta) + (1-p)\beta$

So $P_{Y|X=0} = (1-\beta, \beta)$, $P_{Y|X=1} = (\beta, 1-\beta)$: same entropy in both cases, so in particular true for their average $H(Y|X)$.

Thus $I(X:Y) = H(Y) - H(Y|X)$

$= h_2((1-p)(1-\beta) + p\beta) - h(\beta)$

where $h_2(x) = -x \log x - (1-x)\log(1-x)$

BUT $h_2((1-p)(1-\beta) + p\beta) \leq 1$ with equality if $p = \frac{1}{2}$

Thus $\boxed{C(W) = 1 - h(\beta)}$ QED