Homework

Q1)

Algorithm : input $x \leftarrow P$ or Q

$\mathcal{A}$

If $P(x) = Q(x)$:
- output P w.p $1/2$
- Q w.p $1/2$

Else if $P(x) > Q(x)$:
- output P

Else, $P(x) < Q(x)$:
- output Q

$$\mathbb{P}(win) = \mathbb{P}(\mathcal{A}(x) = P \,/\, x \leftarrow P) \cdot \mathbb{P}(x \leftarrow P) + \mathbb{P}(\mathcal{A}(x) = Q \,/\, x \leftarrow Q) \cdot \mathbb{P}(x \leftarrow Q)$$

$$= \frac{1}{2}\left[\mathbb{P}(\mathcal{A}(x) = P / x \leftarrow P) + \mathbb{P}(\mathcal{A}(x) = Q / x \leftarrow Q) \right]$$

Given $x \in R$, let us define $a_x = \mathbb{P}(\mathcal{A}(x) = P)$

We have :

$$\mathbb{P}(win) = \frac{1}{2}\left[\sum_{x \in R} P(x) \cdot a_x + \sum_{x \in R} Q(x) \cdot (1 - a_x) \right]$$

$$= \frac{1}{2}\left[\sum_{x \in R} a_x \cdot [P(x) - Q(x)] + \sum_{x \in R} Q(x) \right]$$

This holds for every $\mathcal{A}$

$$= \frac{1}{2}\left[1 + \sum_{x \in R} a_x \cdot [P(x) - Q(x)] \right]$$

Now, we have in our case $a_x = \begin{cases} 1/2 & \text{if } P(x) = Q(x) \\ 1 & \text{if } P(x) > Q(x) \\ 0 & \text{else} \end{cases}$

So $\sum_{r \in R} a_r \cdot [P(r) - Q(r)] = \sum_{\substack{r \in R \\ P(r) > Q(r)}} P(r) - Q(r)$

$= \Delta(P, Q)$ by the proposition

So $\mathbb{P}(win) = \frac{1}{2}[1 + \Delta(P, Q)]$

Optimality: Take an algorithm $\mathcal{A}$, and as before let $a_r = \mathbb{P}(\mathcal{A}(r) = P)$, we also have:

$$\mathbb{P}(win) = \frac{1}{2}\left[1 + \sum_{r \in R} a_r \cdot [P(r) - Q(r)]\right]$$

Now let us bound

delete negative terms

$$\sum_{r \in R} a_r \cdot [P(r) - Q(r)] \leq \sum_{\substack{r \in R \\ P(r) > Q(r)}} \underbrace{a_r}_{\leq 1} [P(r) - Q(r)]$$

$$\leq \sum_{\substack{r \in R \\ P(r) > Q(r)}} P(r) - Q(r) = \Delta(P, Q)$$

So $\mathbb{P}(win) \leq \frac{1}{2}[1 + \Delta(P, Q)]$ $\square$

Q2) • $\rho_1 = |+X+|$ is already diagonalized of eigen values 1 and 0 so

$$\rho_1 \log(\rho_1) = 1 \cdot \log(1) |+X+| = 0$$

hence $H(\rho_1) = 0$

• $\rho_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}|0X0| + \frac{1}{2}|1X1|$

So $\rho_2 \log(\rho_2) = -\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

and $H(\rho_2) = 1$

Exercise 2

Let M hermitian $\qquad M = \rho - \rho'$

1. Write $M = \sum_{i=1}^{n} \lambda_i |\psi_i\rangle\langle\psi_i|$

With $|\psi_i\rangle$ O.N. basis, we have that

$$UMU^{\dagger} = \sum_{i=1}^{n} \lambda_i \, U|\psi_i\rangle (U|\psi_i\rangle)^{\dagger}$$

with $U|\psi_i\rangle$ O.N.B. So M and $UMU^{\dagger}$ share the same eigenvalues, hence

$$\mathrm{Tr}(|UMU^{\dagger}|) = \mathrm{Tr}(U|M|U^{\dagger}) = \mathrm{Tr}(|M|) \quad \square$$

2. Let ρ_1, ρ_2, ρ_3 density matrix

$$\Delta(\rho_1, \rho_2) = \frac{1}{2}\mathrm{Tr}(\Pi(\rho_1 - \rho_2)) \text{ for a certain } \Pi$$

$$= \frac{1}{2}\mathrm{Tr}(\Pi(\rho_1 - \rho_3)) + \frac{1}{2}\mathrm{Tr}(\Pi(\rho_3 - \rho_2))$$

$$\leq \Delta(\rho_1, \rho_3) + \Delta(\rho_3, \rho_2)$$

3. a) $\Pi = \sum_{i=1}^{r} |a_i\rangle\langle a_i| \qquad M = \sum_{j=1}^{n} \lambda_j |b_j\rangle\langle b_j|$

$$\Pi \cdot M = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_j |b_j\rangle\langle b_j | a_i\rangle\langle a_i|$$

$$= \sum_{i,j} \lambda_j \cdot \langle b_j | a_i\rangle \, |b_j\rangle\langle a_i|$$

$$\mathrm{Tr}(\Pi \cdot M) = \sum_{ij} \lambda_j \langle b_j | a_i \rangle \langle a_i | b_j \rangle$$

$$= \sum_{ij} \lambda_j |\langle a_i | b_j \rangle|^2$$

$$= \sum_j \lambda_j \left(\sum_i |\langle a_i | b_j \rangle|^2 \right) \rightarrow \leq \||b_j\rangle\|^2 = 1$$

$$\leq \sum_j \lambda_j = \mathrm{Tr}(M)$$

b) Write $M = \sum_{i=1}^{n} \lambda_i |\psi_i \rangle\langle \psi_i|$

$\lambda_1, \dots, \lambda_a \geq 0$

$\lambda_{a+1}, \dots, \lambda_n < 0$

$$= \sum_{i=1}^{a} \lambda_i |\psi_i\rangle\langle\psi_i| - \sum_{i=a+1}^{n} (-\lambda_i) |\psi_i\rangle\langle\psi_i|$$

$$= A - B$$

And $|M| = \sum_{i=1}^{n} |\lambda_i| \, |\psi_i\rangle\langle\psi_i|$

$$= \sum_{i \leq a} \lambda_i |\psi_i\rangle\langle\psi_i| + \sum_{i \geq a+1} (-\lambda_i) |\psi_i\rangle\langle\psi_i|$$

$$= A + B$$

c) Let $M = \rho - \rho' = A - B$ Note that

$\mathrm{Tr}(\rho) = \mathrm{Tr}(\rho') = 1$ so $\mathrm{Tr}(M) = 0 \Rightarrow \mathrm{Tr}(A) = \mathrm{Tr}(B)$

$\Delta(\rho, \rho')$ $\frac{1}{2}\mathrm{Tr}(|\rho - \rho'|) = \frac{1}{2}\mathrm{Tr}(A + B) = \frac{1}{2}(\mathrm{Tr}(A) + \mathrm{Tr}(B))$

$= \mathrm{Tr}(A)$

Φ is a quantum channel, so it is trace preserving

$$\frac{1}{2}\mathrm{Tr}(|\rho - \rho'|) = \mathrm{Tr}(\Phi(A))$$

Now let Π / $\Delta(\phi(\rho), \phi(\rho')) = \mathrm{Tr}\left[\Pi(\phi(\rho) - \phi(\rho'))\right]$

We have $\operatorname{Tr}(\phi(A)) \geqslant \operatorname{Tr}(\Pi \cdot \phi(A))$

$$\geqslant \operatorname{Tr}(\Pi \cdot [\phi(A) - \phi(B)])$$

$$= \operatorname{Tr}(\Pi(\phi(A-B)))$$

$$= \operatorname{Tr}(\Pi \phi(\rho - \rho')) = \Delta(\phi(\rho), \phi(\rho'))$$

$\square$

TUTORIAL 11

1 Homework 8

1. Let P, Q be two probability distributions over a set R. An element r is sampled from P with probability $1/2$ and from Q with probability $1/2$.
   (a) Proprose an algorithm which, on input r, distinguish between the case $r \leftarrow P$ and $r \leftarrow Q$ with probability $1/2 + 1/2 \cdot \Delta(P, Q)$.
   (b) Show that this success probability is optimal.
2. Compute $H(\rho)$ for:
   (a) $\rho = |+\rangle\langle+|$.
   (b) $\rho = I/2$.

2 Trace distance through a quantum channel

We recall that the trace distance is defined as follows:

Definition 2.1. $\Delta(\rho, \rho') = 1/2 \cdot \mathrm{Tr}\,(|\rho - \rho'|) = \min_\pi \mathrm{Tr}\,(\pi(\rho - \rho'))$ *where the minimum is taken among the orthogonal projectors.*

1. Prove that $\Delta(\rho, \rho') = \Delta(U\rho U^\dagger, U\rho' U^\dagger)$ for any ρ, ρ' density operator and U unitary.
2. Show that the trace distance follows the triangular inequality.
3. Let ρ, ρ' be two density operators. Let Φ a quantum channel. We are going to show that quantum channels can only make the trace distance decrease.
   (a) Let M a positive hermitian operator and π an orthogonal projector. Show that $\mathrm{Tr}\,(\pi M) \leq \mathrm{Tr}\,(M)$.
   (b) Let M a hermitian operator. Show that $M = A - B$ with A, B positive hermitian. Show that in this case, $|M| = A + B$.
   (c) Prove that $\Delta(\rho, \rho') \geq \Delta(\Phi(\rho), \Phi(\rho'))$.

3 Information Theory Quantities

3.1 Conditional entropy

Definition 3.1. *The conditional entropy $H(X|Y)$ is defined by*

$$H(X|Y) = \sum_{y \in \mathcal{Y}} P_Y(y) H(X|Y = y) ,$$

where $H(X|Y = y)$ is the entropy of the conditional distribution $P_{X|Y=y}$. Note that elements $y \in \mathcal{Y}$ with $P_Y(y) = 0$ do not participate to the sum.

1. What is the value of $H(X|X)$?

   A: *$H(X|X = x) = 0$ since $P_{X|X=x}$ is a dirac. Thus, $H(X|X) = 0$.*

2. If X and Y are independent random variables, what is the value of $H(X|Y)$?

   A: *$H(X|Y) = \sum_{y \in \mathcal{Y}} P_Y(y) H(P_{X|Y=y}) = \sum_{y \in \mathcal{Y}} P_Y(y) H(P_X) = H(X)$.*

3. Prove that $0 \leq H(X|Y) \leq \log_2(|\mathcal{X}|)$.

   A: *For all y, $0 \leq H(X|Y = y) \leq \log_2(|\mathcal{X}|)$, and since $H(X|Y)$ is a convex combination of the $H(X|Y = y)$, we get the claimed result.*

4. Prove that $H(X|Y) = H(XY) - H(Y)$, where $H(XY) := H((X,Y)) = -\sum_{x,y} P_{XY}(x,y) \log_2 P_{XY}(x,y)$.

   A: *This is just writing $P_{X|Y=y}(x) = \frac{P_{XY}(x,y)}{P_Y(y)}$ in terms of entropies. We have*

$$
\begin{aligned}
H(XY) &= -\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{XY}(x,y) \log_2(P_{XY}(x,y)) \\
&= -\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{XY}(x,y) \log_2(P_Y(y) P_{X|Y=y}(x)) \\
&= -\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{XY}(x,y) \log_2 P_Y(y) - \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{XY}(x,y) \log_2 P_{X|Y=y}(x) \\
&= -\sum_{y \in \mathcal{Y}} P_Y(y) \log_2 P_Y(y) - \sum_{y \in \mathcal{Y}} P_Y(y) \sum_{x \in \mathcal{X}} P_{X|Y=y}(x) \log_2 P_{X|Y=y}(x) \\
&= H(Y) + H(X|Y) \, .
\end{aligned}
$$

3.2 Mutual Information

Definition 3.2. *The mutual information is defined by*

$$
\begin{aligned}
I(X:Y) &= H(X) - H(X|Y) \\
&= H(X) + H(Y) - H(XY) \, .
\end{aligned}
$$

Writing out the definitions, we get $I(X:Y) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{XY}(x,y) \log_2 \frac{P_{XY}(x,y)}{P_X(x) P_Y(y)}$.

1. What is the value of $I(X:X)$?

   A: *$I(X:X) = H(X) - H(X|X) = H(X)$.*

2. If X and Y are independent, what is the value of $I(X:Y)$?

   A: *$I(X:Y) = H(X) - H(X|Y) = H(X) - H(X) = 0$.*

3. For any pair of random variables, prove that $I(X:Y) \geq 0$.
   Hint: Use Jensen's inequality on $-I(X:Y)$.

A:

$$
\begin{aligned}
-I(X:Y) &= -\sum_{x\in\mathcal{X},y\in\mathcal{Y}} P_{XY}(x,y)\log_2\frac{P_{XY}(x,y)}{P_X(x)P_Y(y)} \\
&= \sum_{x\in\mathcal{X},y\in\mathcal{Y}} P_{XY}(x,y)\log_2\frac{P_X(x)P_Y(y)}{P_{XY}(x,y)} \\
&\leq \log_2\left(\sum_{x\in\mathcal{X},y\in\mathcal{Y}} P_{XY}(x,y)\frac{P_X(x)P_Y(y)}{P_{XY}(x,y)}\right) \\
&= \log_2\left(\sum_{x\in\mathcal{X},y\in\mathcal{Y}} P_X(x)P_Y(y)\right) = \log_2(1) = 0
\end{aligned}
$$

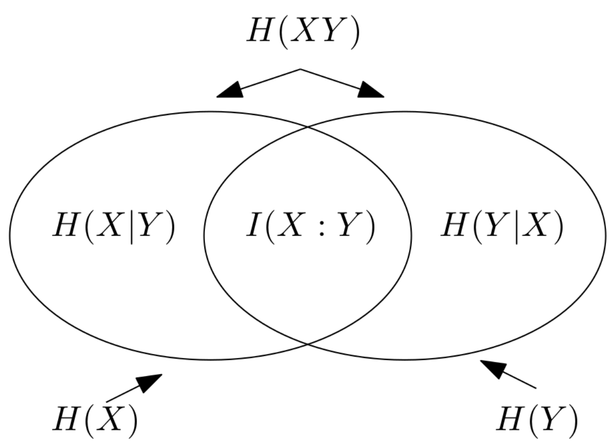

Figure 1: Relation between the entropic measures we have introduced