TUTORIAL 10

1 Homework 7

1. Compute the Kraus representation of the following circuit:
   (a) Add an ancilla qubit $|0\rangle$.
   (b) Apply the 2-qubit operator U.
   (c) Take the partial trace over the first qubit.
2. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$. Let Φ be the phase flip channel with probability $p = \frac{1}{2}$. Compute $\Phi(M)$.

2 Partial Trace

1. We first consider the space of two qubits $\mathbb{C}^2_A \otimes \mathbb{C}^2_B$, A denoting the first qubit and B the second:
   (a) Write the pure state $|\phi_1\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ as a density matrix ρ^1. Compute $\rho^1_A = \mathrm{Tr}_B(\rho^1)$.
   (b) Write the mixture of state $|++\rangle$ with probability $\frac{1}{2}$ and $|--\rangle$ with probability $\frac{1}{2}$ as a density matrix ρ^2. Compute $\rho^2_B = \mathrm{Tr}_A(\rho^1)$.
   (c) Consider $\rho = \frac{1}{12}\begin{pmatrix} 5 & 1 & -1 & 3 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 5 \end{pmatrix}$. What properties would you need on ρ to prove that it is a density matrix? Compute $\rho_A = \mathrm{Tr}_B(\rho)$.
2. From now on, we consider a density matrix ρ_{AB} on an unknown bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$:
   (a) Show that $\rho_A := \mathrm{Tr}_B(\rho_{AB})$ is a valid density matrix.
   (b) Suppose ρ_{AB} is a pure state, ie. $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$ for $|\psi\rangle_{AB} = \sum_{j=1}^{d_A}\sum_{k=1}^{d_B} C_{j,k}|b_j\rangle_A \otimes |c_k\rangle_B$, with $\{|b_j\rangle_A\}_j$ and $\{|c_k\rangle_B\}_k$ orthonormal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$. Let C be the $d_A \times d_B$ matrix with entries $C_{j,k}$.
   Show that $\rho_A = CC^\dagger$ and $\rho_B = C^\dagger C$.
   (c) Consider a classical probability distribution P_{XY}. Calculate the marginal distribution P_X for:
   $$P_{XY}(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y) = (0,0), \\ \frac{1}{2} & \text{if } (x,y) = (1,1), \\ 0 & \text{otherwise,} \end{cases}$$
   with alphabets $\mathcal{X} = \mathcal{Y} = \{0,1\}$. How can we represent P_{XY} in the form of a density matrix? Compute the partial trace of P_{XY} in its quantum representation.

3 Bloch Ball and Sphere

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$ be a matrix. We recall the following facts:

- If $A = A^\dagger$ then A is diagonalizable in an orthonormal basis and its eigenvalues are real.
- If A is diagonalizable and λ_-, λ_+ are its eigenvalues, then $\mathrm{Tr}\,(A) = \lambda_- + \lambda_+$, $\det(A) = \lambda_-\lambda_+$.

We recall that the **Pauli matrices** $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The **Bloch Ball** to be the set of operator of the form

$$\rho(\mathbf{r}) = \frac{1}{2}\left(I + r_X \cdot \sigma_X + r_Y \cdot \sigma_Y + r_Z \cdot \sigma_Z\right) = \frac{1}{2}\left(I + \mathbf{r} \cdot \sigma\right).$$

for $\|\mathbf{r}\| \leq 1$. The **Bloch Sphere** is the set of $\rho(\mathbf{r})$ for $\|\mathbf{r}\| = 1$.

1. Diagonalize the states represented by the Bloch vectors $(1/2, 0, 0)$ and $(1/\sqrt{2}, 0, 1/\sqrt{2})$.
2. Show that any operator in the Bloch Ball is a density operator.
3. Show the converse: that any density operator is in the Bloch Ball.
4. Show that the Bloch Sphere is exactly the set of pure states.

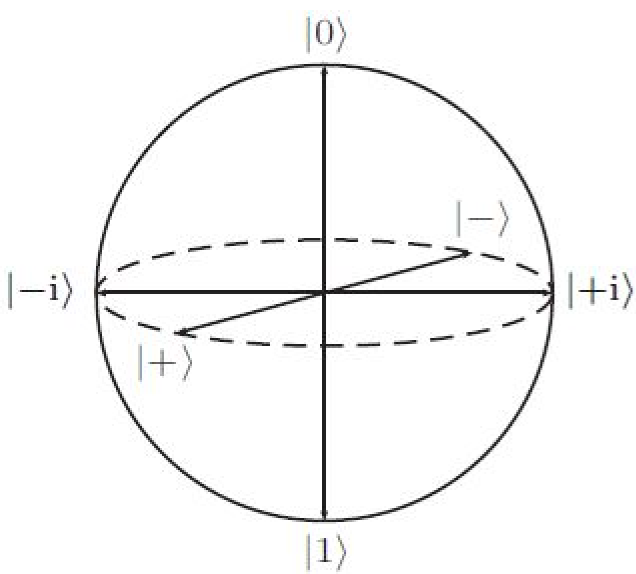

Figure 1: The Bloch Sphere