HW 7

1.

(a) $\{ I \otimes |0\rangle \}$

(b) $\{ U \}$

(c) $\{ \langle 0| \otimes I, \quad \langle 1| \otimes I \}$

Overall: $\{ (\langle 0| \otimes I) U (I \otimes |0\rangle),$
$(\langle 1| \otimes I) U (I \otimes |0\rangle) \}$

2. We have $\Phi(\rho) = p \cdot Z \rho Z^{\dagger} + (1-p) \cdot \rho$
$= p Z \rho Z + (1-p) \cdot \rho$

So as $Z \cdot M = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$

$Z \cdot M \cdot Z = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$

$\Phi(M) = \begin{pmatrix} a & (1-2p)b \\ (1-2p)c & d \end{pmatrix}$

Ex 3

1. • $\rho(\frac{1}{2}, 0, 0) = \frac{1}{2}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$

We see that $\rho(1/2, 0, 0) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}$

$= \frac{3}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So eigenvalues : $\frac{3}{4}$ and $1/4$

We see $\rho(1/2, 0, 0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

we have $\rho(\frac{1}{2}, 0, 0) = \frac{3}{4} \cdot |+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|$

• $\rho(\frac{1}{\sqrt{2}}, 0, 1/\sqrt{2}) = \frac{1}{2}(I + H)$

eigenvalues of H : ± 1

$I + H$: $\{0, 2\}$

$\frac{1}{2}(I + H)$: $\{0, 1\}$

Eigenvectors : the same as H : $\begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}$

2. $\rho(\vec{r}) = \frac{1}{2}\begin{bmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{bmatrix}$ hermitian ok

$Tr = 1$ ok

$Det = \frac{1}{4}\left[1 - r_z^2 - r_x^2 - r_y^2\right]$

$= \frac{1}{4}\left[1 - \|\vec{r}\|^2\right] \geq 0$

$\Rightarrow$ eigenvalues ≥ 0

3. Just write it

4. Bloch Sphere : $\|\vec{r}\| = 1$ $\Rightarrow$ $Det = 0$

$\Rightarrow Tr = 1$

So $\exists |\psi\rangle$ s.t. $\rho(\vec{r}) = |\psi\rangle\langle\psi|$

$+ 0(\ldots)$

TD 10

2 Partial Trace

1 $\rho^1 = |\phi_1\rangle\langle\phi_1| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$$\rho^1_A = Tr_B(\rho^1) = \sum_{\substack{b \text{ basis} \\ \text{of } B}} \mathbb{1}_A \otimes \langle b| \rho_A \, \mathbb{1}_A \otimes |b\rangle$$

$$= \mathbb{1}_A \otimes \langle 0|_B \rho^1 \mathbb{1}_A \otimes |0\rangle_B + \mathbb{1}_A \otimes \langle 1|_B \rho^1 \mathbb{1}_A \otimes |1\rangle_B$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \quad 0) \, \rho^1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} +$$

Always true for 2 qubits

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rho^1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rho^1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{\mathbb{1}_A}{2}$$

2 $\rho^2 = \frac{1}{2}|++\rangle\langle++| + \frac{1}{2}|--\rangle\langle--|$

<u>But</u> $|++\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$|--\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$

So $|++\rangle\langle++| = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $|--\rangle\langle--| = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

So $\rho^2 = \frac{1}{2}|++\rangle\langle++| + \frac{1}{2}|--\rangle\langle--| = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

We have that $\rho_B^2 = \mathrm{Tr}_A(\rho^2) = (1\ 0)\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\rho^2\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$+ (0\ 1)\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}\rho^2\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$$= \begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}\rho^2\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0&1&0\\0&0&0&1\end{pmatrix}\rho^2\begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1&0\\0&1\\0&1\\1&0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\\1&0\\0&1\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$= \frac{\mathbb{1}_B}{2}$$

(c) Density matrix: • $\mathrm{Tr}\,\rho = \frac{5+1+1+5}{12} = 1$ ok

• $\rho^\dagger = \rho$ ok

• $\rho \geq 0$: Find all eigenvalues, check that ≥ 0
(enough since ρ diagonalizable since Hermiti

$$\rho_A = \begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}\rho\begin{pmatrix}1&0\\0&0\\0&1\\0&0\end{pmatrix} + \begin{pmatrix}0&1&0&0\\0&0&0&1\end{pmatrix}\rho\begin{pmatrix}0&0\\1&0\\0&0\\0&1\end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}\begin{pmatrix}5&-1\\1&-1\\-1&1\\3&1\end{pmatrix} + \frac{1}{12}\begin{pmatrix}0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&3\\1&-1\\-1&1\\-1&5\end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix}5&-1\\-1&1\end{pmatrix} + \frac{1}{12}\begin{pmatrix}1&-1\\-1&5\end{pmatrix} = \frac{1}{6}\begin{pmatrix}3&-1\\-1&3\end{pmatrix}$$

2 (a) First $\rho_{AB} = \sum_{jklm} p_{jl,km} |b_j\rangle\langle b_k|_A \otimes |c_l\rangle\langle c_m|_B$

Then $\rho_A = \sum_{jk} \left(\sum_l p_{jl,kl} \right) |b_j\rangle\langle b_k|_A \quad \left(= \sum_l \mathbb{1}_A \otimes \langle c_l| \rho\, \mathbb{1}_A \otimes |c_l\rangle \right)$

→ Since $\rho_{AB}^\dagger = \rho_{AB}$, we have $p_{jl,km} = \overline{p_{km,jl}}$

thus $\sum_l p_{jl,kl} = \sum_l \overline{p_{kl,jl}} = \overline{\sum_l p_{kl,jl}}$

so $\rho_A^\dagger = \rho_A$.

→ $\mathrm{Tr}\,\rho_A = \sum_j \left(\sum_l p_{jl,jl} \right) = \sum_{jl} p_{jl,jl} = \mathrm{Tr}\,\rho_{AB} = 1$

→ Consider $|\psi\rangle_A \otimes |c_l\rangle_B$ for some l
arbitrary

then $\langle\psi| \otimes \langle c_l|\, \rho_{AB}\, |\psi\rangle \otimes |c_l\rangle \geq 0$ since $\rho_{AB} \succeq 0$

So $\underbrace{\sum_l \langle\psi| \otimes \langle c_l| \rho_{AB} |\psi\rangle \otimes |c_l\rangle \geq 0}$

$\langle\psi| \left(\underbrace{\sum_l \mathbb{1}_A \otimes \langle c_l|_B\, \rho_{AB}\, | \mathbb{1}_A \otimes c_l\rangle_B}_{= \rho_A} \right) |\psi\rangle = \langle\psi|\rho_A|\psi\rangle$

so $\langle\psi|\rho_A|\psi\rangle \geq 0$ for any state $|\psi\rangle$ so $\boxed{\rho_A \succeq 0}$

(b) $\rho_{AB} = \sum_{j\ell j'\ell'} c_{j\ell}\overline{c_{j'\ell'}}\, |b_j\rangle\langle b_{j'}| \otimes |c_\ell\rangle\langle c_{\ell'}|$

So $\rho_A = \mathrm{Tr}_B\, \rho_{AB} = \sum_{jj'} \Big(\underbrace{\sum_\ell c_{j\ell}\overline{c_{j'\ell}}}_{= [CC^\dagger]_{jj'}}\Big) |b_j\rangle\langle b_{j'}|$

and the ρ_B case is symmetric

(c). $P_X(0) = P_{XY}(0,0) + P_{XY}(0,1) = \frac{1}{2} + 0$

$P_X(1) = 1 - P_X(0) = \frac{1}{2}$

$\Rightarrow \boxed{P_X(x) = \frac{1}{2}}$

. $P_{XY} \rightsquigarrow \rho_{XY} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$

diagonal density matrix !

Then $P_X \rightsquigarrow \rho_X = \mathrm{Tr}_Y(\rho_{XY}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$