# TD 1

## 1 - HW1

1. Apply H and measure
2. If $v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $v' = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$, then $v \otimes v' = \begin{pmatrix} \alpha v' \\ \beta v' \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$

If $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = v \otimes v' = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$, then $\alpha\gamma = \beta\delta = \frac{1}{\sqrt{2}}$

so $\alpha, \beta, \gamma, \delta \neq 0$ But $\alpha\delta = 0$ so $\alpha = 0$ or $\delta = 0$.

This is a contradiction. Thus $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ is not a product state.

## 2 - Quantum Random Access Code

1. A classical random access code is a function $f : \{0,1\}^2 \to \{0,1\}$ and a decoding strategy (possibly randomized) which, given the choice of the first or second bit, should decode correctly $x_1$ (resp. $x_2$) given $f(x_1, x_2)$ with probability at least $p$ for all $x_1, x_2 \in \{0,1\}$

Note that the probability is taken only on the decoding strategy, and should succeed with probability at least $p$ for all possible inputs $x_1, x_2$.

An example: $f$: $00 \mapsto 0$, $01 \mapsto 1$, $10 \mapsto 0$, $11 \mapsto 1$ : for the second bit, given $b = f(x_1, x_2)$ we output $b$, which is $x_2$ in all cases, so we succeed with prob. 1

for the first bit, we do not have any information.
The best we can do to succeed in <u>all cases</u> with the best probability is to output uniformly 0 or 1. Then we succeed with probability 0.5 for all inputs.

<u>The proof</u>: $f : \underbrace{\{0,1\}^2}_{\text{size } 4} \to \underbrace{\{0,1\}}_{\text{size } 2}$ $\quad$ size 4 $>$ size 2

So, there exists two couples $(x_1, x_2) \neq (x_1', x_2')$ such that
$f(x_1 x_2) = f(x_1' x_2') = b$.

Thus, either $x_1 \neq x_1'$ or $x_2 \neq x_2'$. Let us assume we are in the first case, and that $x_1 = 0$ and $x_1' = 1$.

We are asked to guess the first bit given $b = f(x_1 x_2) = f(x_1' x_2')$

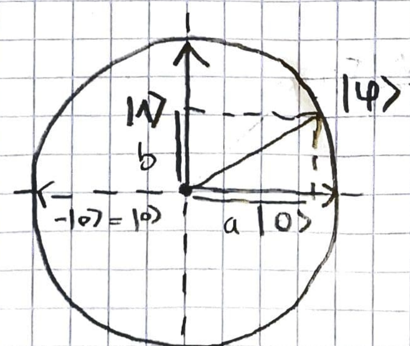
The possible strategies are only of the form: $\boxed{\begin{array}{l} \text{Output } 1 \text{ wp } p \\ \quad\quad\quad\;\; 0 \text{ wp } 1-p \end{array}}$

<u>with $p$ depending on $b$.</u>

Thus, in our case, this strategy decides $x_1 = 0$ wp $1-p$
and $x_1' = 1$ wp $p$

So it has a global probability of success of at most
$\min(p, 1-p) \leq 0.5$, and the statement is proved.

2. A real qubit is a unit vector of the circle. More precisely, it's even the projective line crossing the center of the circle, since the global phase of the qubit does not matter:




- To measure, we project the qubit $|\psi\rangle$ on the basis $(|0\rangle, |1\rangle)$ and take the square of its norm.

$|\psi\rangle$ outputs $|0\rangle$ wp $|a|^2$ when measured
$\quad\quad\quad\quad\quad\;\; |1\rangle$ wp $|b|^2$

with $a = \langle 0|\psi\rangle$
$\quad\quad\; b = \langle 1|\psi\rangle$

In 2-dimensions, the only real unitaries are rotations and reflexions.

- We can "measure" in fact in any orthonormal basis, not only $(|0\rangle, |1\rangle)$

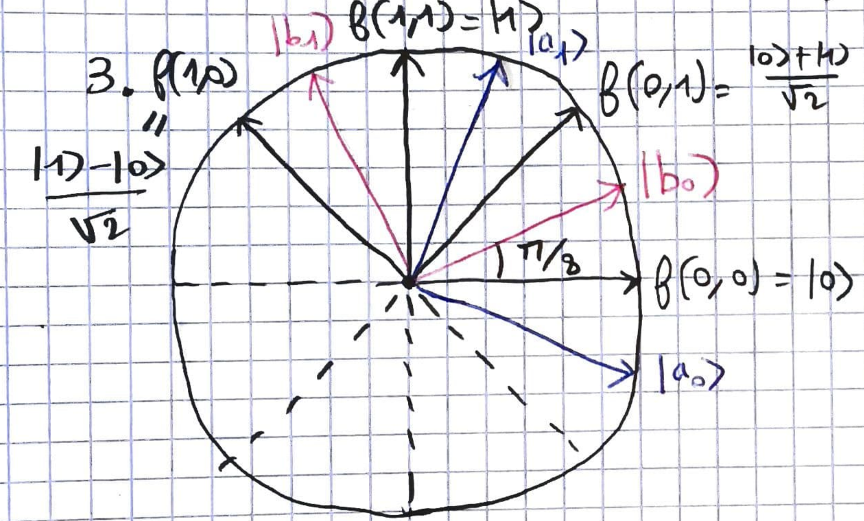
Indeed, if we want to measure in $(|b_1\rangle, |b_2\rangle)$, then $U = [b_1 | b_2]$ is a unitary. ( $\langle b_i | b_i \rangle = 1$ and $\langle b_1 | b_2 \rangle = 0$ )

Since $\langle b_i | \varphi \rangle = \langle i | U^\dagger | \varphi \rangle = \langle i | ( U^\dagger | \varphi \rangle )$ and $U |i\rangle = |b_i\rangle$,

we apply $U^\dagger$ on $|\varphi\rangle$, measure on $(|0\rangle, |1\rangle)$, and apply $U$ on the result

$\Rightarrow$ this is the same as "measuring" $|\varphi\rangle$ on $(|b_0\rangle, |b_1\rangle)$.

3.




Take $U_1 = (b_0 | b_1)^\dagger$
$U_2 = (a_0 | a_1)^\dagger$
in the definition
is the same as measuring in $(|b_0\rangle, |b_1\rangle)$ for decode the first bit and in $(|a_0\rangle, |a_1\rangle)$ to decode the second one.

Indeed: • $\mathbb{P}(\text{Output} = x_1 \mid 1^{\text{st}} \text{ bit measured}) = |\langle x_1 | U_1 | f(x_1 x_2) \rangle|^2$

$= |(U_1^\dagger |x_1\rangle)^\dagger | f(x_1 x_2) \rangle|^2 = |\langle b_{x_1} | f(x_1 x_2) \rangle|^2$

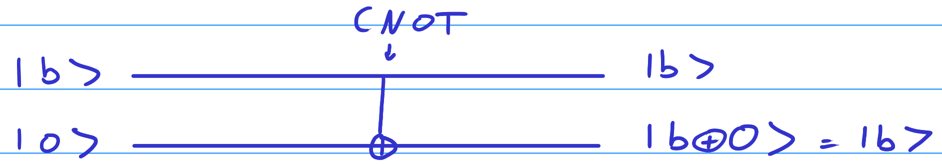
• Similarly, $\mathbb{P}(\text{Output} = x_2 \mid 2^{\text{nd}} \text{ bit measured}) = |\langle a_{x_2} | f(x_1 x_2) \rangle|^2$

In all 4 cases in each case, one can check that there is always a $\pi/8$ angle, so we get always $p = \cos^2\left(\frac{\pi}{8}\right) \simeq 0.85$

QED.

# Exercise 3

1.




2. $\text{CNOT} \cdot \left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \right)$

$$= \frac{1}{\sqrt{2}} \left[ \text{CNOT}\, |00\rangle + \text{CNOT}\, |10\rangle \right] \quad \text{by linearity}$$

$$= \frac{1}{\sqrt{2}} \left[ |00\rangle + |11\rangle \right]$$

The qubit $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $(= |+\rangle)$ is not cloned since

$$\text{CNOT}\, |+\rangle \neq |+\rangle \otimes |+\rangle = \frac{1}{2}\left[ |00\rangle + |01\rangle + |10\rangle + |11\rangle \right]$$

3. Assume that $\exists\, U$ unitary such that for any qubit $|\psi\rangle$, we have $U \cdot |\psi\rangle \otimes |0\rangle = |\psi\rangle|\psi\rangle$.

Then take $|\psi\rangle, |\varphi\rangle$ such that $\langle \psi | \varphi \rangle \notin \{0,1\}$ (for example take $|\psi\rangle = |0\rangle$ and $|\varphi\rangle = |+\rangle$, so that $\langle \psi | \varphi \rangle = 1/\sqrt{2}$), we have that

$$\underbrace{(|\psi\rangle|0\rangle) \cdot (|\varphi\rangle|0\rangle)}_{\text{scalar product}} = \langle 0 | \langle \psi | \varphi \rangle |0\rangle$$

$$= \langle 0 | 0 \rangle \cdot \langle \psi | \varphi \rangle$$

$$= \langle \psi | \varphi \rangle$$

On the other hand, as $U^\dagger \cdot U = I$,

$$\langle 0 | \langle \psi | \varphi \rangle |0\rangle = \langle 0 | \langle \psi | U^\dagger U | \varphi \rangle |0\rangle$$

$$= \left[ U |\psi\rangle|0\rangle \right]^\dagger \cdot \left[ U |\varphi\rangle|0\rangle \right]$$

$$= \left[ |\psi\rangle|\psi\rangle \right]^\dagger \cdot \left[ |\varphi\rangle|\varphi\rangle \right]$$

$$= \langle \psi | \langle \psi | \varphi \rangle | \varphi \rangle$$

$$= \langle \psi | \varphi \rangle^2$$

So we have $\langle \psi | \varphi \rangle^2 = \langle \psi | \varphi \rangle$, which is a contradiction $\square$